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THE RATIONAL LOEWY SERIES AND NILPOTENT IDEALS OF ENDOMORPHISM RINGS

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ABSTRACT

Sufficient conditions are given, in module-theoretic terms, for the ideal $N(S)$ of the endomorphism ring S of a module M consisting of the endomorphisms with essential kernel to be nilpotent. This extends in a natural way several known results on the nilpotency of $N(S)$. When M is a quasi-injective module such that S is right noetherian, it is shown that S is right artinian if and only if M has a finite rational Loewy series whose length is, in this case, equal to the index of nilpotency of $N(S)$.

Let M be a left R-module and $S = \mathrm{End}({}_RM)$. Then the set $N(S)$ of all the endomorphisms of M which have essential kernel is a two-sided ideal of S. The problem of giving criterions for the nilpotency of $N(S)$ (and bounds on the index of nilpotency) is interesting for several reasons. For instance, if M is a quasi-injective module, then $N(S)$ is the Jacobson radical of S, and if M is a self-faithful quasi-projective module [10], then $N(S)$ coincides with the left singular ideal of S. On the other hand, if M is a finite-dimensional module, then as it was shown in [7] and [16], the nilpotency of $N(S)$ is a sufficient condition for the nil subrings of S to be nilpotent. This last property was used by Shock [16] to prove that if a module M satisfies the ascending chain condition on rationally closed submodules, then nil subrings of S are nilpotent. A more recent source of ideas on this problem is [3] where R is supposed to be a ring with Krull dimension and the main tool used is the concept of semicritical socle series. Later, it was shown in [11] and [19] how these methods could be extended to torsionfree modules with respect to a torsion theory τ of the

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category R-mod of left R-modules. The basic assumption in [19] is that R is a τ-artinian ring and, in order to obtain a bound on the index of nilpotency of $N(S)$, M is furthermore supposed to be uniform and quasi-injective.

In this paper we give, in Theorem 1.2, sufficient conditions in module-theoretic terms for $N(S)$ to be nilpotent and we use this to get a result (Corollary 1.6) which can be regarded as a sort of dual of the theorem of Shock mentioned above. As particular cases of our theorem we recover several results of [1], [6], [19] and [20] whose basis is the fact that M is a Δ-module (i.e., R satisfies the descending chain condition (DCC) on annihilators of subsets of M) but, since in this case no particular hypotheses are placed on the ring R, the result is applicable also in more general situations (when M is not necessarily a Δ-module). Our main tool is the concept of (ascending) rational Loewy series of M, $\{F^\alpha(M)\}$. This is the particular case of the F_τ-series of M considered in [11] obtained when $\tau = \chi(M)$ is the torsion theory of R-mod cogenerated by M and, in some cases, it coincides with the $\chi(M)$-semicocritical socle series of M in the sense of [19].

In the second part of the paper we consider the question of giving (in terms of the rational Loewy series) bounds for the index of nilpotency of $N(S)$ and in Proposition 2.1 we obtain a module-theoretic extension of [19, Theorem 4.8] without assuming that M is uniform nor quasi-injective. We can be much more precise when M is a quasi-injective module. In this case we show in Theorem 2.2 that $F^1(M)$ is the annihilator in M of the radical J of S and if, furthermore, M satisfies an additional condition (which holds, e.g., when S is right noetherian), then $F^n(M)$ is the annihilator of J^n for each $n \geqq 0$. Thus, for these modules, J is nilpotent with index n precisely when M has a rational Loewy series of length n and, in particular, for a quasi-injective module M, S is right artinian if and only if M has DCC on (rationally) closed submodules and a finite rational Loewy series. If, in addition to the above hypotheses, M is supposed to be uniform, then we show that J^n is, for each $n \geqq 0$, the annihilator in S of $F^n(M)$.

Throughout this paper R denotes an associative ring with identity and R-mod the category of left R-modules. A module X is said to be L-generated if it is a quotient of a direct sum of copies of L and L-cogenerated if it is a submodule of a direct product of copies of L. The largest L-generated submodule of module X (the trace of L in X) is denoted by X_L. The class of all the modules which are isomorphic to submodules of M-generated modules is denoted by $\sigma[M]$. This class has been considered among other papers, in [8] (where the notation $\overline{\mathrm{Gen}}({}_RM)$ is used) and [21]. The module M is called quasi-injective when every homomorphism from a submodule of M to M has an extension to M; the dual concept is that of a quasi-projective module. We use the notation $E(M)$ $(\tilde{M})$ to stand for an injective (quasi-injective) hull of M [5].

If τ is a (hereditary) torsion theory of R-mod, the lattice of τ-closed submodules of a module M consists of all the $X \subset M$ such that M/X is τ-torsionfree. When M has DCC on τ-closed submodules, M is said to be τ-artinian. A submodule X of M is called τ-dense if M/X is a τ-torsion module and a nonzero module C is τ-cocritical when it is τ-torsionfree and every proper submodule of C is τ-dense. We refer the reader to [2], [5], [11] and [17] for all the ring-theoretic and torsion-theoretic notions used in the text.

1. Endomorphism rings of distinguished modules

Let M and X be left R-modules. As in [16], a submodule Y of X is said to be M-rational in X (or X a M-rational extension of Y) when every homomorphism f from a submodule of X to M such that Ker f contains Y must be the zero map. Also, a submodule Y of X is called M-rationally closed if Y has no proper rational èxtensions in X (when $X = M$ we will supress the prefix M-). Thus, the M-rational submodules of X are precisely the $\chi(M)$-dense submodules and the M-rationally closed submodules are the $\chi(M)$-closed submodules. The M-rationally closed sumbodules of X form a complete modular lattice ([12], [17]) and we will denote by Y^c the smallest M-rationally closed submodule which contains a given submodule Y. This lattice is also pseudo-complemented (for it is isomorphic to the lattice of subobjects of an object of a Grothendieck category [17, Coroll. IX.4.4]) and hence it is complemented if and only if it has no proper essential elements. Then it is easy to see that the property of having a complemented lattice of M-rationally closed submodules is preserved under direct sums, submodules and quotients and by [17, Coroll. VI.1.8] we have the following lemma.

LEMMA 1.1. *Each R-module X has a largest submodule with complemented lattice of M-rationally closed submodules*, $F_M(X)$. *Moreover*, F_M *defines a left exact subfunctor of the identity of R*-mod.

The construction of F_M is carried out (in a slightly different way) in [11], where a left exact subfunctor F_τ of the identity of R-mod is associated to each torsion theory τ. Taking $\tau = \chi(M)$ in this construction we obtain precisely F_M. Observe also that, in the particular case that X is $E(M)$-cogenerated (i.e., X is a

$\chi(M)$-torsionfree module), then $F_M(X)$ is just the largest submodule Y of X such that each essential submodule of Y is M-rational in Y. In [11], an ascending Loewy series is associated to the left exact subfunctor of the identity F_τ in the usual manner; we will be interested in the following particular case. For each R-module M set

$$F^0(M) = 0, \quad F^\alpha(M)/F^{\alpha-1}(M) = F_M(M/F^{\alpha-1}(M))$$

if α is a non-limit ordinal and $F^\alpha(M) = \bigcup_{\beta<\alpha} F^\beta(M)$ if α is a limit ordinal ($F^1(M)$ will also be denoted by $F(M)$). Then $\{F^\alpha(M)\}$ will be called the (ascending) rational Loewy series of M. There exists a least ordinal γ such that $F^\gamma(M) = F^{\gamma+1}(M)$ and when $\gamma = n$ is finite, we say that M has finite rational Loewy length. If, furthermore, $F^n(M) = M$, then we will say that M has a finite rational Loewy series (of length n).

Recall that if M and L are R-modules, then M is called L-distinguished [15] when for each nonzero homomorphism $f: X \to M$, there exists a homomorphism $g: L \to X$ such that $f \cdot g \neq 0$. If this condition holds for all nonzero homomorphisms $f: X \to M$ where X is a module of $\sigma[M]$, then we will say that M is L-distinguished in $\sigma[M]$ (it may happen that M be L-distinguished in $\sigma[M]$ without being L-distinguished). We will make use of this concept to get a module theoretic extension of [19, Theorem 3.2] which, in particular, gives the nilpotency of $N(S)$. Calling $H = \mathrm{Hom}_R(L, \tilde{M})$ we will say as in [1] and [5] that a submodule Z of L is $\tilde{M}$-closed if, with the notation of [1], $Z = l_L r_H(Z)$, that is, Z is equal to the intersection of the kernels of all the endomorphisms $f: L \to \tilde{M}$ such that $f(Z) = 0$. This is also equivalent to L/Z being cogenerated by $\tilde{M}$. Note also that if $H' = \mathrm{Hom}_R(N, E(M))$, then $Z^c = l_L r_{H'}(Z)$ and, in particular, a $\tilde{M}$-closed submodule of L is M-rationally closed (the converse is not true in general).

THEOREM 1.2. *Let M be a left R-module and $S = \mathrm{End}({}_RM)$. If there exists an R-module L such that M is L-distinguished in $\sigma[M]$ and L satisfies the DCC on $\tilde{M}$-closed submodules, then M has a finite rational Loewy series and $N(S)$ is nilpotent.*

PROOF. Let $\{F^\alpha(M)\}$ be the rational Loewy series of M. We claim that $F^\alpha(M) = M$ for some ordinal α. Using a standard argument which essentially reduces to [17, Prop. VI.2.5], we see that to prove this it is enough to show that each nonzero $\chi(M)$-torsionfree quotient module of M contains a nonzero submodule Y with the property that each essential submodule of Y is M-rational in Y. Since $\chi(M)$-cocritical modules have obviously this property, this will follow if we show, more generally, that each nonzero $\chi(M)$-torsionfree module of $\sigma[M]$ contains a $\chi(M)$-cocritical submodule. Let then X be a $\chi(M)$-torsionfree module of $\sigma[M]$. Then there exists a nonzero homomorphism $f: X \to E(M)$ which induces another nonzero homomorphism $g: f^{-1}(M) \to M$. But $f^{-1}(M)$, being a submodule of X, belongs to $\sigma[M]$ and it follows from the fact that M is L-distinguished in $\sigma[M]$ that there exists a homomorphism $h: L \to f^{-1}(M)$ such that $g \cdot h \neq 0$. Let now $Y \subset \operatorname{Im} h$ be a M-rationally closed submodule of $\operatorname{Im} h$. Then Y is equal to the intersection of the kernels of the homomorphisms $u: \operatorname{Im} h \to E(M)$ such that $Y \subset \operatorname{Ker} u$. But if u is such a homomorphism we see, using the fact that $\operatorname{Im} h \subset X$ is a module of $\sigma[M]$ and the injectivity of $E(M)$, that there exists a homomorphism $t: M^{(I)} \to E(M)$ (for some set I) such that $\operatorname{Im} u \subset \operatorname{Im} t$. As it is well known (see, e.g. [6, Prop. 5.1A]), the quasi-injective hull $\tilde{M}$ of M is precisely $E(M)_M$ and hence we have that $\operatorname{Im} u \subset \operatorname{Im} t \subset E(M)_M = \tilde{M}$. Thus Y is $\tilde{M}$-closed in $\operatorname{Im} h$ and so $h^{-1}(Y)$ is a $\tilde{M}$-closed submodule of L. Since L has, by hypothesis, DCC on such submodules, we see that $\operatorname{Im} h$ has DCC on M-rationally closed submodules. If C is a minimal M-rationally closed submodule of $\operatorname{Im} h$, then C is obviously $\chi(M)$-cocritical (for $\operatorname{Im} h$ is $\chi(M)$-torsionfree) and $C \subset \operatorname{Im} h \subset X$, proving our claim.

Next we show that the rational Loewy series of M is finite. Consider for each $n \geqq 1$, the submodule

$$L_n = \bigcap \{X \subset L \mid X \text{ is } \tilde{M}\text{-closed}, F_M^n(L/X) = L/X\}$$

(where F_M^n denotes the left exact subfunctor of the identity corresponding to the n-th term of the ascending Loewy series associated with F_M). We have a descending chain $L_1 \supset L_2 \supset \cdots \supset L_i \supset \cdots$ of $\tilde{M}$-closed submodules of L which, by hypothesis, stabilizes, say at $L_r = L_{r+1}$. Now we claim that $F^r(M) = M$. To prove this consider an epimorphism $p: L^{(J)} \to (F^{r+1}(M))_L$ for some set J and let $i_j: L \to L^{(J)}$, $q_j: L_r \to L_r^{(J)}$ be the canonical injections and $v: L_r^{(J)} \to L^{(J)}$ the inclusion. Since F_M^{r+1} is a left exact subfunctor of the identity we see that $F_M^{r+1}(F^{r+1}(M)) = F^{r+1}(M)$ and hence that $F_M^{r+1}(\operatorname{Im}(p \cdot i_j)) = \operatorname{Im}(p \cdot i_j)$, so that $L_r = L_{r+1} \subset \operatorname{Ker}(p \cdot i_j)$ for each j of J. Therefore, we have that $p \cdot v \cdot q_j = 0$ for each j, which means that $p \cdot v = 0$ and hence that there exists an epimorphism $k: (L/L_r)^{(J)} \to (F^{r+1}(M))_L$. On the other hand, since L has DCC on $\tilde{M}$-closed submodules, we have that there exists a finite set $\{X_1, \ldots, X_n\}$ of $\tilde{M}$-closed submodules of L such that $F_M^r(L/X_i) = L/X_i$ for each $i = 1, \ldots, n$ and $L_r = \bigcap_1^n X_i$. Thus we have a monomorphism $L/L_r \to \oplus_1^n L/X_i$ and, since $F_M^r(\oplus L/X_i) = \oplus F^r(L/X_i) = \oplus L/X_i$, we get that $F_M^r(L/L_r) = L/L_r$. Since subfunctors of the identity preserve arbitrary direct sums we also have that

$F_M^r((L/L_r)^{(J)}) = (L/L_r)^{(J)}$ and, from the existence of the above-mentioned epimorphism k, it follows that $F_M^r(F^{r+1}(M))_L = (F^{r+1}(M))_L$, which in turn implies that $F_M^r(F^{r+1}(M))$ contains $(F^{r+1}(M))_L$. If we set $Z = F^{r+1}(M)/F_M^r(F^{r+1}(M))$ and there is a nonzero homomorphism $z: Z \to E(M)$, then we get a nonzero homomorphism $s: F^{r+1}(M) \to E(M)$ which induces $0 \neq c: s^{-1}(M) \to M$. Since $s^{-1}(M)$ belongs to $\sigma[M]$ and M is L-distinguished in $\sigma[M]$, there is a homomorphism $d: L \to s^{-1}(M)$ such that $c \cdot d \neq 0$. But this contradicts the fact that $\operatorname{Im} d \subset (F^{r+1}(M))_L \subset F_M^r(F^{r+1}(M))$. Therefore we must have $z = 0$, which means that $F^{r+1}(M)$ is a M-rational extension of $F_M^r(F^{r+1}(M))$. Since, obviously, $F_M^r(F^{r+1}(M))$ is a M-rationally closed submodule of $F^{r+1}(M)$, we see that, in fact, $F_M^r(F^{r+1}(M)) = F^{r+1}(M)$, from which it follows that $F^{r+1}(M) = F^r(M)$ and hence that $F^r(M) = M$. To complete the proof of the theorem, observe that, by [11, Prop. 2.2], $F^r(M)N(S)^r = 0$ and hence, in this case, $N(S)^r = 0$.

In the course of the above proof we have shown that in the hypotheses of Theorem 1 each nonzero $\chi(M)$-torsionfree module of $\sigma[M]$ contains a $\chi(M)$-cocritical submodule. From this it is easy to see that in that case $F(M)$ coincides with the $\chi(M)$-socle of M (see [12] for the definition) and also that $F^n(M)$ is equal to the n-th term of the $\chi(M)$-semicocritical socle series of M defined in [19] (as we will see later, this does not hold in general).

From Theorem 1.2 it follows that if $S = \operatorname{End}(_RM)$ and there exists an R-module L such that M is L-distinguished and L has DCC on M-rationally closed submodules, then $N(S)$ is nilpotent. A particular case of this situation is:

Corollary 1.3 [20, Coroll. 6.2]. *Let τ be a torsion theory of R-*mod *such that R is τ-artinian and M is a τ-torsionfree module with $S = \operatorname{End}(_RM)$. Then $N(S)$ is nilpotent.*

Proof. Since M is τ-torsionfree and R τ-artinian, it is clear that R has DCC on M-rationally closed left ideals and so the result follows from Theorem 1.2.

Corollary 1.4 [6, Coroll. 8.3]. *If M is a quasi-injective Δ-module and $S = \operatorname{End}(_RM)$, then S has nilpotent radical.*

Proof. Applying Theorem 1.2 to $L = {_RR}$ we see that $N(S)$ is nilpotent. On the other hand, since M is quasi-injective, it is well known that $N(S)$ is precisely the radical of S.

If M is a quasi-injective module and X a submodule of M, then it follows from the fact that M is fully invariant in its injective hull [5, Coroll. 19.3] that X is M-closed (i.e., $X = l_M r_S(X)$) if and only if it is a rationally closed submodule and we will say simply that X is a closed submodule. The following result extends [1, Prop. 9.10] and [6, Coroll. 6.4].

COROLLARY 1.5. *Let ${}_RM$ be a quasi-injective Δ-module with DCC on closed submodules. Then $S = \operatorname{End}({}_RM)$ is right artinian. This happens, in particular, when M is a quasi-injective Δ-module such that there exists a finitely generated submodule X of M with $r_S(X) = 0$.*

PROOF. By Corollary 1.4, S has nilpotent radical. Furthermore, if $X \subset M$, then it is clear that X is an essential submodule of $l_M r_S(X)$ and hence each essentially closed submodule of M is closed. Therefore M has DCC on essentially closed submodules and this is equivalent to M being finite-dimensional [13, Theorem 3.14]. This implies that S is semiperfect (see, e.g. [17, Prop. XIV.1.7]) and hence semiprimary. On the other hand, the DCC on closed submodules of M is equivalent to S being right noetherian [1, Coroll. 4.3] and thus we get that S is right artinian. To prove the second statement, let $\bar{R} = R/l_R(M)$. Then M is an injective $\bar{R}$-module [1, Prop. 2.16] and if τ is the torsion theory of $\bar{R}$-mod cogenerated by M, our hypothesis implies that $\bar{R}$ is τ-artinian. Since $r_S(X) = 0$, we have that $l_M r_S(X) = M$ and so X is a $\bar{R}$-finitely generated τ-dense submodule of M. Thus it follows from [12, Coroll. 21.4] that M is also τ-artinian, that is, M has DCC on closed submodules, which completes the proof.

A module M which is M-distinguished in $\sigma[M]$ has been called M-faithful in [10] (we will also say that M is self-faithful in this case). In [16, Theorem 10] it is shown that if M has ACC on rationally closed submodules, then $N(S)$ in nilpotent and every nil subring of S is nilpotent. Similarly we have:

COROLLARY 1.6. *Let M be a self-faithful module with DCC on rationally closed submodules. Then $N(S)$ is nilpotent and nil subrings of S are nilpotent. If, furthermore, M is quasi-projective, then the left singular ideal of S is nilpotent.*

PROOF. $N(S)$ is nilpotent by Theorem 1.2 and that nil subrings of S are nilpotent is a consequence of the fact that in this case M is finite-dimensional, using [16, Theorem 3]. The last assertion follows from the fact that, if M is self-faithful and quasi-projective, then $N(S)$ is precisely the left singular ideal of S by [9, Coroll. 4.4].

As a consequence of Corollary 1.6 we see that if R is a ring with DCC on rationally closed left ideals, then the left singular ideal Z of R is nilpotent. We remark that the DCC on left annihilators is not enough to guarantee the nilpotency of Z [14]. Note also that, if in addition to the hypotheses of Corollary 1.6 M is supposed to be quasi-injective, then we may see as in the proof of Corollary 1.5 that S is, in fact, right artinian. In the following result we make the DCC hypothesis of L independent of M.

COROLLARY 1.7. *Let L be an R-module with DCC on L-generated submodules and M an L-distinguished module with $S = \mathrm{End}({}_RM)$. Then $N(S)$ is nilpotent.*

PROOF. Using Theorem 1.2 it will be enough to show that L has DCC on M-rationally closed submodules. Let X be a submodule of L and assume that there is a nonzero homomorphism $f: X/X_L \to E(M)$. If $p: X \to X/X_L$ denotes the canonical projection we have that, since $E(M)$ is L-distinguished by [15, Theorem 1], there exists a homomorphism $g: L \to X$ such that $f \cdot p \cdot g \neq 0$. But clearly $p \cdot g = 0$ and this contradiction shows that f must be zero, i.e., that X_L is M-rational in X. If X is furthermore M-rationally closed in L, then X is the M-rational closure $(X_L)^c$ of X_L in L and so $X \to X_L$ gives an order-preserving injective mapping from the set of M-rationally closed submodules of L to the set of L-generated submodules, from which the result follows.

COROLLARY 1.8. *Let ${}_RP$ be a projective module with DCC on P-generated submodules and $\{S_i\}_I$ a set of representatives of the isomorphism classes of the simple quotients of ${}_RP$. If M is cogenerated by the $E(S_i)$, then $N(S)$ is nilpotent.*

PROOF. This follows from Corollary 1.7 observing that M is P-distinguished.

EXAMPLES 1.9. Unlike the modules which satisfy the hypotheses of Corollaries 1,3, 1.4 and 1.5, the modules M of Corollaries 1.6, 1.7 and 1.8 need not be Δ-modules, even in the case that they are furthermore supposed to be injective or (finitely generated) projective. For instance, in Cozzens' example of a non-artinian simple noetherian domain with a unique simple module C which is injective [5, p. 90], C satisfies trivially all the conditions of Corollary 1.6 but it is not a Δ-module.

For the projective case, let A be a ring with DCC on rationally closed left ideals (i.e., a Δ-ring in the terminology of [6]) and $R = RFM(A)$ the ring of row-finite $N \times N$ matrices over A. Let $e = e_{11}$ and $P = Re$. Then P is a finitely generated projective left ideal of R which may be identified with A_A^N and if $p_i : P \rightarrow A$ denotes the i-th projection and ${}_RX$ is a nonzero submodule of ${}_RP$, then there exists a left ideal $I \neq 0$ of A such that $p_i(X) = I$ for each i of N (see [8, Example 4.5]). Thus, if $0 \neq X \subset {}_RP$, there exists an element x of X such that $p_1(x) \neq 0$ and considering the homomorphism $f: {}_RP \rightarrow X$ given by right multiplication with x we have that $f(e) = ex = p_1(x) \neq 0$, so that $f \neq 0$ and hence P is self-faithful (or, equivalently, P-distinguished). To see that P has DCC on rationally closed submodules, let $E(A)$ be the injective hull of ${}_AA$ and consider the (A, R)-bimodule ${}_AeR_R$. Since eR_R is projective, the left R-module $E = \mathrm{Hom}_A(eR, E(A))$ is injective. E can be identified with the set of $N \times N$ matrices (a_{ij}) such that $a_{i1} \in E(A)$ and $a_{ij} = 0$ for $j > 1$, and it is clear that ${}_RP$ is an essential submodule of E, so that in fact $E = E({}_RP)$. Observe now that the rationally closed submodules of ${}_RP$ are precisely the annihilators in P (P considered as a left ideal of R) of the subsets of E. If $Z \subset E$, its annihilator in P can be identified with $(l_A(\pi_1(Z)))^N$ (where $\pi_1 : E(A)^N \rightarrow E(A)$ denotes the first projection and we identify E with $E(A)^N$) and since A is a Δ-ring we see that ${}_RP$ has indeed DCC on rationally closed submodules. Finally, observe that P is not a Δ-module, for the left ideal of the matrices of R that have all the entries in the first n columns zero is the annihilator in R of a subset of P for each $n \geqq 0$.

If we assume that the ring A is left artinian, then since $\mathrm{End}({}_RP) \simeq eRe \simeq A$ we have by [9, Prop. 3.1] that ${}_RP$ is an artinian module and hence we may apply Corollary 1.8 to see that if ${}_RM$ is cogenerated by the injective envelopes of the simple quotients of ${}_RP$ and $T = \mathrm{End}({}_RM)$, then $N(T)$ is nilpotent.

The fact that, when A is a Δ-ring in the above example, then $S = \mathrm{End}({}_RP)$ has $N(S)$ nilpotent, follows also from the remarks made after Corollary 1.6, for $Z({}_AA)$ is nilpotent. More generally, we have:

Corollary 1.10. *Let R be a ring such that there exists a module ${}_RL$ with DCC on R-rationally closed submodules and with $L_R^* = \mathrm{Hom}_R(L, R)$ faithful. Then $Z({}_RR)$ is nilpotent.*

Proof. By [15, Example 2], L_R^* is faithful if and only if ${}_RR$ is L-distinguished and so the result follows from Theorem 1.2.

Recall from [22] that a module L is called a Σ-self-generator if all the submodules of L^r are L-generated (for each $r \geqq 0$).

Corollary 1.11. *Let L be a Σ-self-generator and $M \in \sigma[L]$ such that L*

has DCC on $\tilde{M}$-closed submodules. Then $N(S)$ is nilpotent. If, furthermore, L is artinian, then $N(S)$ is nilpotent for every M of $\sigma[L]$.

PROOF. It is easy to see that if L is a Σ-self-generator, then each module of $\sigma[L]$ is L-generated. If M belongs to $\sigma[L]$, we have that $\sigma[M] \subset \sigma[L]$ and it is clear that M is L-distinguished in $\sigma[M]$, so that we may apply Theorem 1.2. The last assertion is now immediate.

We recall that a module ${}_RM$ is called counterartinian [6] when M_S is artinian.

COROLLARY 1.12. *Let ${}_RP$ be a projective counterartinian module and $B = \mathrm{Biend}({}_RP)$ its biendomorphism ring. Then $N(B)$ is a nilpotent ideal.*

PROOF. By [12, Theorem 3.2], P_S is a Σ-self-generator and thus the result follows from Corollary 1.11.

One might naturally ask what happens if in Corollary 1.6 the assumption that M is self-faithful is dropped. In this case $N(S)$ is not necessarily nilpotent (for instance, take $M = Z_{p}\infty$) but we can still show that nil subrings of S are nilpotent.

PROPOSITION 1.13. *Let M be an R-module with DCC on rationally closed submodules and $S = \mathrm{End}({}_RM)$. Then each nil subring of S is nilpotent.*

PROOF. We adapt the nice proof given by Fisher for artinian modules [7, Theorem 1.5]. First we show that S has DCC on left annihilators (of subsets of S). If $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$ is a descending chain of left annilators in S, then we have a descending chain of rationally closed submodules of $M: (MI_1)^c \supset (MI_2)^c \supset \cdots$ and so $(MI_j)^c = (MI_{j+1})^c$ for some j. Now, if $f \in r_S(I_{j+1})$, then it is clear that $MI_{j+1} \subset \mathrm{Ker}\, f$ and since $\mathrm{Ker}\, f$ is a rationally closed submodule of M, $(MI_{j+1})^c$ is also contained in $\mathrm{Ker}\, f$, so that $MI_j \subset \mathrm{Ker}\, f$ and $f \in r_S(I_j)$. Therefore $r_S(I_j) = r_S(I_{j+1})$ and hence $I_j = I_{j+1}$. Let now N be a nil subring of S. By [7, Lemma 1.2], in order to prove that N is nilpotent it is enough to show that it is right T-nilpotent. Assume, on the contrary, that N is not right T-nilpotent. Then, by [7, Lemma 1.3], there exists a sequence $\{\phi_n\}$ of elements of N such that $\phi_n\phi_{n-1}\cdots\phi_1 = 0$ and $\phi_i\phi_{n-1}\cdots\phi_1 = 0$ for each $n \geqq i$. Calling $s_j = \phi_j\phi_{j-1}\cdots\phi_1$ for each j, we see that $Ms_1 \supset Ms_2 \supset \cdots \supset Ms_j \supset \cdots$ and hence we have a descending chain of rationally closed submodules of M, $\{(Ms_j)^c\}$. Our hypothesis implies that $(Ms_k)^c = (Ms_{k+1})^c$ for some k and this means that Ms_k is a M-rational extension of Ms_{k+1}. We claim that this implies that $\mathrm{Ker}\, s_i + \mathrm{Im}\, \phi_{i+1}$ is a rational submodule of M for each $i \geqq k$. By

[18, Lemma 1.1], to prove this it is enough to show that if $x \in M$, $0 \neq y \in M$, then there exists $r \in R$ such that $rx \in \operatorname{Ker} s_i + \operatorname{Im} \phi_{i+1}$, $ry \neq 0$. By a similar argument we have that, since $s_{i+1}(M)$ is M-rational in $s_i(M)$, there exists $r \in R$ such that $rs_1(x) \in s_{i+1}(M)$ and $ry \neq 0$. Therefore $rs_i(x) = s_{i+1}(z)$ for some $z \in M$ and so $rx = (rx - \phi_{i+1}(z)) + \phi_{i+1}(z)$ with $s_i(rx - \phi_{i+1}(z)) = 0$. Thus we have indeed shown that $M = (\operatorname{Ker} s_i + \operatorname{Im} \phi_{i+1})^c$ and, since for $i \geqq k$, $\operatorname{Im} \phi_{i+1} \subset \operatorname{Ker} s_n$ for each $n > i$, we have also that $M = (\operatorname{Ker} s_i + \bigcap_{n>i} \operatorname{Ker} s_n)^c$ for each $i \geqq k$. Observe now that, since the lattice of rationally closed submodules of M is modular, we have as in [7, Lemma 1.4] that if X, Y, Z are rationally closed submodules of M such that $X \subset Y$, $(X+Z)^c = (Y+Z)^c$ and $X \cap Z = Y \cap Z$, then $Y = Y \cap (Y+Z)^c = Y \cap (X+Z)^c = (X + (Y \cap Z))^c = (X + (X \cap Z))^c = X^c = X$. Applying this recursively to $X_t = \bigcap_{r=1}^{t+1} \operatorname{Ker} s_{k+r}$, $Y_t = \bigcap_{r=1}^{t} \operatorname{Ker} s_{k+r}$ (with $Y_0 = M$) and $Z_t = \operatorname{Ker} s_k$ for $t = 0, 1, \ldots,$ we see that $\operatorname{Ker} s_k \supset \operatorname{Ker} s_k \cap \operatorname{Ker} s_{k+1} \supset \cdots$ is a strictly descending chain of rationally closed submodules of M, which gives a contradiction and completes the proof.

In [4] it is shown that if M is the rational completion of a noetherian module and $S = \operatorname{End}({}_RM)$, then nil subrings of S are nilpotent. Similarly we have:

Corollary 1.14. *Let ${}_RM$ be a rational extension of an artinian module and $S = \operatorname{End}({}_RM)$. Then nil subrings of S are nilpotent.*

Proof. Let L be an artinian rational submodule of M. Then the lattice of rationally closed submodules of M is clearly isomorphic to the lattice of rationally closed submodules of L and hence M has DCC on rationally closed submodules.

2. Endomorphism rings of quasi-injective modules and bounds on the index of nilpotency

We have observed in the proof of Theorem 1.2 that if M has a finite rational Loewy series of length n and $S = \operatorname{End}({}_RM)$, then n is an upper bound for the index of nilpotency of $N(S)$ but we are going to obtain a more precise bound. In order to do this we say (in a similar way to [19]) that a submodule X of L is linked to M at the i-th layer if there exists a nonzero submodule Y of $F^i(M)/F^{i-1}(M)$ such that

$$X \subset l_L(Y) = \cap \{\operatorname{Ker} f \mid f \in \operatorname{Hom}_R(L, Y)\}.$$

The next result extends [19, Theorem 4.8], which is obtained taking $L = R$ a ring with DCC on M-rationally closed left ideals and assuming that M is, furthermore, uniform and quasi-injective.

PROPOSITION 2.1. *Let M and L be left R-modules such that M is L-distinguished in $\sigma[M]$ and L has DCC on $\tilde{M}$-closed submodules. If*

$$D = l_L(F(M)) = \bigcap\{\operatorname{Ker} f \mid f \in \operatorname{Hom}_R(L, F(M))\} \quad \text{and} \quad S = \operatorname{End}({}_R M),$$

then the index of nilpotency of $N(S)$ is less than or equal to the number of layers to which D is linked.

PROOF. We use induction in the number k of layers to which D is linked. By definition, D is always linked to $F(M)$ (assuming $M \neq 0$), so that our induction starts for $k = 1$ assuming that D is not linked to the i-th layer for $i > 1$. If $f \in N(S)$, then $\operatorname{Ker} f$ is essential and rationally closed in M so that $F(M) \subset \operatorname{Ker} f$ by [11, Prop. 2.1]. If $f \neq 0$, there must be a least $i > 1$ such that $f(F^i(M)) \neq 0$ and since $f(F^{i-1}(M)) = 0$, calling $M_i = F^i(M)/F^{i-1}(M)$ we get a nonzero homomorphism $g: M_i \rightarrow M$ induced by f. The proof of Theorem 1.2 shows that every nonzero $\chi(M)$-torsionfree module of $\sigma[M]$ contains a $\chi(M)$-cocritical module, so that the sum of all the $\chi(M)$-cocritical submodules of M_i is essential and hence M-rational in M_i (for $M_i = F_M(M/F^{i-1}(M))$). Therefore, there must be a $\chi(M)$-cocritical submodule C of M_i such that $g(C) \neq 0$ for, on the contrary, $\operatorname{Ker} g$ would be a M-rational submodule of M_i in contradiction with the fact that $g \neq 0$. Then $C \simeq g(C) \subset g(M_i) \subset F(M)$, so that D annihilates C. This contradicts the fact that D is not linked to the i-th layer and thus we have in this case that $N(S) = 0$.

Assume now that the result is true for modules Z such that Z satisfies our hypotheses and $l_L(F(Z))$ is linked to less than k layers of Z. Suppose that D is linked to exactly k layers of M and let the largest of them be the q-th layer. Set $Z = F^{q-1}(M)$. It is clear that Z is L-distinguished in $\sigma[Z]$ and also that L has DCC on $\tilde{Z}$-closed submodules. Moreover, Z is essential in M by [17, Coroll. VI.3.5] and hence $F(Z) = F_Z(Z) = F_M(Z) = F(M)$, and, more generally, $F^i(Z) = F^i(M)$ for each $i \leqq q - 1$. Thus $l_L(F(Z)) = D$ and the layers of Z are the $F^i(M)/F^{i-1}(M)$ for $i = 1, \ldots, q - 1$, so that D is linked to exactly $k - 1$ layers of Z. By the induction hypothesis we have that if $S' = \operatorname{End}({}_R Z)$,then $N(S')^{k-1} = 0$. Now, arguing as in the proof of [19, Theorem 4.8] we have that if $f_1, \ldots, f_k \in N(S)$, then $f_i\,|_Z \in N(S')$ and hence $Z = F^{q-1}(M) \subset \operatorname{Ker}(f_k f_{k-1} \cdots f_2)$. This gives a homomorphism $M_q = F^q(M)/F^{q-1}(M) \rightarrow M$ and, since the lattice of M-rationally closed submodules of M_q is complemented, we have that $(f_k f_{k-1} \cdots f_2)(F^q(M)) \subset F(M)$. Since $f_1(F(M)) = 0$ we also see that $(f_kf_{k-1}\cdots f_1)(F^q(M))=0$. Now, if $f_k\cdots f_1\neq 0$, there exists $m\geqq q$ such that $(f_k\cdots f_1)(F^m(M))=0$ but $(f_k\cdots f_1)(F^{m+1}(M))\neq 0$. This produces a nonzero homomorphism $F^{m+1}(M)/F^m(M)\to M$ and the argument used in the proof of the case $k=1$ shows that D is linked to the $(m+1)$-th layer of M, which is a contradiction and completes the proof.

Let ${}_RM$ be a quasi-injective module and $S=\mathrm{End}({}_RM)$. Then $N(S)$ is precisely the Jacobson radical J of S [2, Prop. 18.20] and we are going to show that, in some cases, the index of nilpotency of J coincides with the length of the rational Loewy series of M.

Theorem 2.2. *Let ${}_RM$ be a quasi-injective module. Then $F(M)=l_M(J)$. If moreover $M/F^n(M)$ contains an essential finite direct sum of $\chi(M)$-cocritical submodules for each $n\geqq 1$, then $F^n(M)=l_M(J^n)$. This happens, in particular, when M (or $M/F(M)$) has DCC on (M-rationally) closed submodules.*

Proof. We have already observed that $F^n(M)J^n=0$ for each $n\geqq 0$, so that, in particular, $F(M)\subset l_M(J)$. On the other hand we know that $F(M)$ is the largest submodule of M with the property that all its essential submodules are M-rational and hence, to prove the converse inclusion, we will show that $l_M(J)$ enjoys this property. Let then K be an essential submodule of $l_M(J)$. We will prove that $\mathrm{Hom}_R(l_M(J)/K, E(M))=0$. To see this, let T be a pseudo-complement of K in M, i.e., T maximal with respect to the property that $K\cap T=0$. Then $K\oplus T$ is essential in M and, using the fact that M, being quasi-injective, is a fully invariant submodule of $E(M)$ [5, Coroll. 19.3], it is straightforward to see that this implies that $l_M(J)\subset(K\oplus T)^c$. Since the join in the lattice of rationally closed submodules of M of X_1^c and X_2^c is given by $(X_1+X_2)^c$, we have that

$$(K+(l_M(J)\cap T))^c=(K^c+(l_M(J)\cap T^c))^c=l_M(J)\cap(K\oplus T)^c,$$

where the last identity follows by modularity and the fact that $K^c\subset l_M(J)$. Since as we have seen $l_M(J)\subset(K\oplus T)^c$, we get that $(K+(l_M(J)\cap T))^c=l_M(J)$. But $K\cap(l_M(J)\cap T)\subset K\cap T=0$ and, as K is by hypothesis essential in $l_M(J)$, $l_M(J)\cap T=0$ and hence $l_M(J)=K^c$. This implies that $\mathrm{Hom}_R(l_M(J)/K, E(M))=0$, completing the proof of the first part.

To prove the second assertion we only have to show that $l_M(J^n)\subset F^n(M)$. Using induction, assume that $n\geqq 2$ and $F^{n-1}(M)=l_M(J^{n-1})$. Then there are homomorphisms $f_i\in J^{n-1}$, $i\in I$, such that $F^{n-1}(M)=\bigcap_I \mathrm{Ker}\, f_i$. If $q:M\to M/F^{n-1}(M)$ denotes the canonical projection, then each f_i factors as $f_i=g_i\cdot q$, with $g_i : M/F^{n-1}(M) \to M$ and $\bigcap_I \operatorname{Ker} g_i = 0$. But $M/F^{n-1}(M)$ contains by hypothesis an essential direct sum of the form $\oplus_1^r C_i$, where the C_i are $\chi(M)$-cocritical modules and an easy induction (see, e.g., the proof of [2, Prop. 10.6, c)$\Rightarrow$a)]) shows that the lattice of M-rationally closed submodules of $\oplus_1^r C_i$ has the finite intersection property (see [2, p. 132] for the definition). Thus it is clear that the lattice of M-rationally closed submodules of $M/F^{n-1}(M)$ has also this property and hence there exists a finite subset $\{i_1, \ldots, i_r\}$ of I such that $\bigcap_{k=1}^r \operatorname{Ker} g_{i_k} = 0$. This means that $F^{n-1}(M) = \bigcap_{k=1}^r \operatorname{Ker} f_{i_k}$ and we have a monomorphism $u : M/F^{n-1}(M) \to M^r$ such that, denoting by $p_k : M^r \to M$ the canonical projections, $f_{i_k} = p_k \cdot u \cdot q$. On the other hand, the argument used above to prove that $F(M) = l_M(J)$ shows also that $F_M(X)$ is the intersection of the kernels of the homomorphisms $f : X \to M$ such that $\operatorname{Ker} f$ is essential, for each $\chi(M)$-torsionfree module X of $\sigma[M]$. This, together with the fact that F_M is a left exact subfunctor of the identity of R-mod, implies that $F_M(M/F^{n-1}(M)) = u^{-1}(F_M(M^r))$ is the intersection of all the kernels of homomorphisms $h : M/F^{n-1}(M) \to M$ which have an extension $v : M^r \to M$ such that $\operatorname{Ker} v$ is essential in M^r (note that, since M is quasi-injective, all homomorphisms from M^r to $E(M)$ factor through M). If h is such a homomorphism, then

$$h \cdot q = v \cdot u \cdot q = v \cdot \left(\sum_1^r j_k \cdot p_k \right) \cdot u \cdot q = \sum_1^r (v \cdot j_k) \cdot (p_k \cdot u \cdot q) = \sum_1^r (v \cdot j_k) \cdot f_{i_k}$$

(where $j_k : M \to M^r$ denote the canonical injections). Since $\operatorname{Ker} v$ is essential in M^r, we have that $\operatorname{Ker}(v \cdot j_k) = j_k^{-1}(\operatorname{Ker} v)$ is essential in M, so that $v \cdot j_k \in J$. Since f_{i_k} belongs to J^{n-1}, we get that $h \cdot q \in J^n$. Now it is clear that

$$F^n(M) = q^{-1}(F_M(M/F^{n-1}(M))) = \bigcap \operatorname{Ker}(h \cdot q)$$

and hence that $l_M(J^n) \subset F^n(M)$. Finally, assume that $M/F(M)$ has DCC on M-rationally closed submodules. Then each $M/F^i(M)$, $i \geqq 1$, is a $\chi(M)$-torsionfree quotient of $M/F(M)$ and it is not difficult to prove that it contains an essential finite direct sum of $\chi(M)$-cocritical submodules (see, e.g., [12, Prop. 18.3, Prop. 21.1]). This ends the proof.

From Theorem 2.2 it follows that if M is a quasi-injective module such that $M/F^n(M)$ is an essential extension of a finite direct sum of $\chi(M)$-cocritical modules for each $n \geqq 1$, then J is nilpotent with index r if and only if M has a finite rational Loewy series of length r. In case M has DCC on closed submodules (i.e., when S is right noetherian) we get the following refinement of Corollary 1.5.

COROLLARY 2.3. *Let ${}_RM$ be a quasi-injective module. Then S is right artinian if and only if ${}_RM$ has DCC on closed submodules and finite rational Loewy series.*

PROOF. We have already remarked that S is right noetherian if and only if M has DCC on closed submodules. As in Corollary 1.5 we have that in this case S is also semiperfect. Bearing in mind that S is right artinian if and only if it is right noetherian and semiprimary, the result follows from Theorem 2.2.

The following corollary extends a well-known result on endomorphism rings of (quasi-)injective modules [2, Coroll. 18.21].

COROLLARY 2.4. *Let M be a quasi-injective module such that $F(M)$ is essential in M. Then $J(S) = r_S(F(M))$ and $S/J(S) \simeq \mathrm{End}(F(M))$.*

Next, we see that, in some cases, the powers of the radical J of S are precisely the annihilators of the corresponding terms of the rational Loewy series.

COROLLARY 2.5. *Let M be a uniform quasi-injective module and $S = \mathrm{End}_R(M)$. If $M/F^n(M)$ is an essential extension of a finite direct sum of $\chi(M)$-cocritical submodules for each $n \geqq 1$, then $J^n = r_S(F^n(M))$. In particular, S is right artinian if and only if M has DCC on closed submodules and finite rational Loewy length.*

PROOF. Since $F^n(M) = l_M(J^n)$ by Theorem 2.2, we have that $J^n \subset r_S l_M(J^n) = r_S(F^n(M))$ and hence we only have to show that $r_S(F^n(M)) \subset J^n$. We do it by induction in n. For $n = 1$, the result follows from Corollary 2.3. Thus, assume that $n \geqq 2$ and $r_S(F^{n-1}(M)) \subset J^{n-1}$ and let $f \in r_S(F^n(M))$. If $p : M \to M/F^{n-1}(M)$, $q_1 : M/F^{n-1}(M) \to M/F^n(M)$ and $q_2 : M \to M/F^n(M)$ denote the canonical homomorphisms, we have that, since $f(F^n(M)) = 0$, there exists $h : M/F^n(M) \to M$ such that $f = h \cdot q_2 = h \cdot q_1 \cdot p$. On the other hand, the proof of Theorem 2.2 shows that $M/F^{n-1}(M)$ embeds in a module of the form M^r and we may assume that r is the least integer with this property. We thus have a monomorphism $u : M/F^{n-1}(M) \to M^r$ and, since M is quasi-injective, there exists a homomorphism $g : M^r \to M$ such that $g \cdot u = h \cdot q_1$ and hence $f = h \cdot q_1 \cdot p = g \cdot u \cdot p$. Therefore, if $p_k : M^r \to M$, $i_k : M \to M^r$, for $k = 1, \ldots, r$, denote the canonical projections and injections, respectively, we have that $p_k \cdot u \cdot p \in r_S(F^{n-1}(M)) \subset J^{n-1}$ and we claim that $g \cdot i_k \in J$ for each $k =$

$1, \ldots, r$. Assume on the contrary that $g \cdot i_k \notin J$ for some k. Since M is uniform, this implies that $\mathrm{Ker}(g \cdot i_k) = 0$. Furthermore, $F_M(M/F^{n-1}(M)) \subset \mathrm{Ker}(g \cdot u)$ and since each $\chi(M)$-cocritical submodule of $M/F^{n-1}(M)$ is contained in $F_M(M/F^{n-1}(M))$, our hypotheses imply that $\mathrm{Ker}(g \cdot u)$ is essential in $M/F^{n-1}(M)$. Thus, if $v : i_k^{-1}(M/F^{n-1}(M)) \to M$ denotes the pull-back of u along i_k, we see that $\mathrm{Ker}\, v = \mathrm{Ker}(g \cdot i_k \cdot v)$ is also essential and, since v is a monomorphism, this implies that $i_k^{-1}(M/F^{n-1}(M)) = 0$. But this means that $M/F^{n-1}(M)$ embeds in M^{r-1}, contradicting the minimality of r and hence $g \cdot i_k \in J$ for each $k = 1, \ldots, r$. Now, it is easy to see that $f = \Sigma_1^r (g \cdot i_k) \cdot (p_k \cdot u \cdot p)$ and thus $f \in J^n$, completing the first part of the proof. To prove the last assertion we must only show that if M has DCC on closed submodules and finite full Loewy length n, then S is right artinian. We have that $F^n(M) = F^{n+1}(M)$ and hence

$$J^n = r_S(F^n(M)) = r_S(F^{n+1}(M)) = J^{n+1}.$$

But, since S is right noetherian, Nakayama's lemma implies that $J^n = 0$ and so that result follows.

EXAMPLES 2.6. There are (uniform) quasi-injective modules M such that $M/F^n(M)$ is a finitely cogenerated module and hence contains an essential finite direct sum of $\chi(M)$-cocritical modules, for each $n \geqq 0$, but $M/F(M)$ has not DCC on M-rationally closed submodules. For instance, if $R = Z_p \ltimes Z_{p^\infty}$ (where p is a prime, Z_p is the ring of p-adic integers and $\ltimes$ denotes the trivial extension) is Osofsky's example of an injective cogenerator ring without chain conditions [5, p. 214], then the rational Loewy series of R coincides with the usual ascending Loewy series and $R/F^n(R) \simeq (p^n, 0)R$ for each $n \geqq 0$. Thus, if J denotes the radical of R and $\mathrm{Soc}(R)$ the socle, $\mathrm{Soc}^n(R) = l_R(J^n)$ and $J^n = r_R(\mathrm{Soc}^n(R))$ for each $n \geqq 0$, but R has not DCC on closed submodules.

Let $C = \mathbf{Z}/p^3\mathbf{Z}$ (p a prime) and R the ring of upper triangular 5×5 matrices with entries $a_{ij} \in C$ for $i < 5$ and $a_{55} \in \mathbf{Z}$. Let τ be the torsion theory of R-mod whose associated Gabriel filter consists of the left ideals of R which contain the ideal of all the $(a_{ij}) \in R$ such that $a_{22} = a_{55} = 0$ and $M = Re_{33}$. In [20, Example 6.13] it is shown that R is a τ-artinian ring and M a τ-torsionfree (uniform) quasi-injective module such that the length of the τ-semicocritical socle series of M is 4 but the index of nilpotency of the radical J of $S = \mathrm{End}(_RM)$ is 3 (actually, S is isomorphic to C). We see that the length of the rational Loewy series of M gives in this case a better bound (in fact, the best possible bound, according to Theorem 2.2). A straightforward computation shows that

$$F(M)=\begin{bmatrix} p^2C \\ p^2C \\ p^2C \\ 0 \\ 0 \end{bmatrix} \quad F^2(M)=\begin{bmatrix} pC \\ pC \\ pC \\ 0 \\ 0 \end{bmatrix} \quad F^3(M)=M=\begin{bmatrix} C \\ C \\ C \\ 0 \\ 0 \end{bmatrix}$$

and this is also the $\chi(M)$-semicocritical socle series of M, which is shorter than other semicocritical socle series (such as the corresponding to the torsion theory τ defined above). From Proposition 2.1 and Theorem 2.2 it follows that in this case $D = l_R(F(M))$ is linked to all the rational layers of M.

Despite the fact that the rational Loewy series of M coincides with the $\chi(M)$-semicocritical socle series in the hypotheses of Theorem 1.2 and Proposition 2.1 as we have already remarked, this is far from being true in general. Perhaps the simplest example is obtained by considering a left self-injective regular ring R which is not a product of left full linear rings (i.e., such that the left socle of R is not essential [13]). Then $\chi(R) = \tau_G$ is the Goldie torsion theory of R-mod ([12]) and, since the τ_G-cocritical modules are the nonsingular uniform left R-modules, which in this case must be simple [13, Prop. 3.24], we have that the τ_G-semicocritical socle of R is $\mathrm{Soc}(R)$ and all the terms of the τ_G-semicocritical socle series are isomorphic to $E(\mathrm{Soc}(R))$. On the other hand, it is clear that $F(R) = R$ and so the ring satisfies trivially the hypotheses of Corollary 2.3 without having essential (τ_G-semicocritical) socle.

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